901. Substituting $y = \frac{1}{6}(8 - 5x)$ gives

$$\begin{aligned} & x^2 + \tfrac{1}{36}(8-5x)^2 = 1 \\ \Longrightarrow\ & 61x^2 - 80x + 28 = 0. \end{aligned}$$

This has $\Delta = 80^2 - 4 \cdot 61 \cdot 28 = -432 < 0$, so there are no simultaneous solution points. In other words, the line $5x + 6y = 8$ doesn't intersect the unit circle.

902. Using the definition $\ln x \equiv \log_e x$ and the log rule $\log x^n \equiv n \log x$, we simplify as follows:

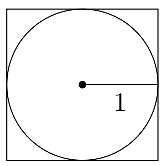
$$\begin{aligned} & \ln \sqrt{e} + \ln \sqrt[3]{e} + \ln \sqrt[6]{e} \\ = & \tfrac{1}{2} \ln e + \tfrac{1}{3} \ln e + \tfrac{1}{6} \ln e \\ = & \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{6} \\ = & 1. \end{aligned}$$

903. Let $\mathrm{g}(x)$ be the best linear approximation to $\mathrm{f}(x)$ at $x = 1$. We need $y = \mathrm{g}(x)$ to be the tangent line to $y = \mathrm{f}(x)$ at $x = 1$. So, we differentiate:

$$\mathrm{f}'(x) = 9x^2 - 5.$$

This gives $\mathrm{f}'(1) = 4$. Passing through $(1, -2)$, the tangent line is $y = 4x - 6$. Hence, close to $x = 1$, $\mathrm{f}(x) \approx 4x - 6$.

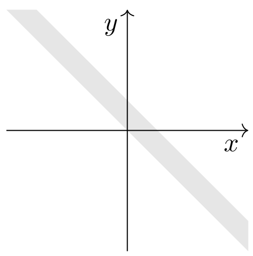
904. In this problem, the square is the possibility space, and area corresponds to probability. So, we need only calculate the relevant areas. Let the radius have length 1:



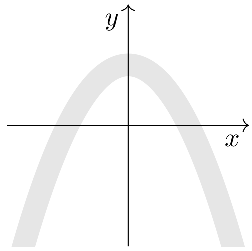
The areas of square and circle are 4 and π. So the probability that a randomly chosen point is inside the circle is $\frac{\pi}{4}$.

905. Correlation is closeness to a linear relationship.

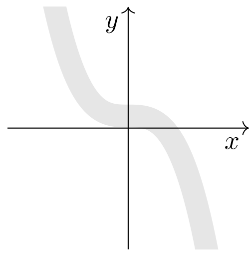
(a) Since $x + y \approx k$, x and y are related linearly, with a negative gradient. A sample is likely to show strong negative correlation.



(b) Since $x + y^2 \approx k$, the relationship is parabolic. And, since $\bar{x} = 0$, values in the sample will be approximately symmetrically distributed around $x = 0$. Since correlation is closeness to a <u>linear</u> relationship, samples are unlikely to show correlation.



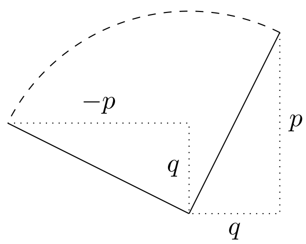
(c) Such a cubic relationship is likely to show some negative correlation, depending on the spread of the sample, but weaker than that in (a). The behaviour of x^3, 3 being odd, resembles that of x^1 (linear) more than that of x^2 does.



906. For a quadratic sequence $Q_n = an^2 + bn + c$, the second difference is $2a$. In this case, the second difference is $(21 - 8) - (8 - 1) = 6$, which gives $a = 3$.

Substituting $n = 1$ and $n = 2$, we get $1 = 3 + b + c$ and $8 = 12 + 2b + c$. Subtracting these, $b = -2$ and then $c = 0$. So, the ordinal nth term formula is $Q_n = 3n^2 - 2n$.

907. In the diagram below, the two triangles are images under rotation, so they are congruent. The values labelled are displacements, i.e. signed lengths.



The gradients are p/q and $q/-p$, which are negative reciprocals. QED.

— Alternative Method —

A rotation by 90°, i.e. rotation to perpendicular, is equivalent to a reflection in $x = 0$ followed by a reflection in $y = x$. Reflection in $x = 0$ *negates* gradients; reflection in $y = x$ switches x and y, which *reciprocates* gradients. In combination, this gives a negative reciprocal. $\square$

908. (a) Differentiating with respect to y,

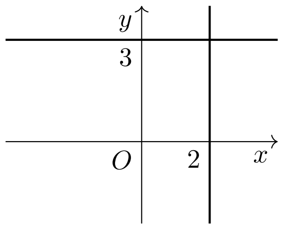
$$\frac{dx}{dy} = 4y.$$

Reciprocating both sides gives $\dfrac{dy}{dx} = \dfrac{1}{4y}$.

(b) At $y = \frac{1}{8}$, the gradient is 2, and $x = \frac{33}{32}$. So, using $y - y_1 = m(x - x_1)$, the tangent has equation

$$\begin{aligned} & y - \tfrac{1}{8} = 2\big(x - \tfrac{33}{32}\big) \\ \Longrightarrow\ & 16y - 2 = 32x - 33 \\ \Longrightarrow\ & 32x = 16y + 31. \end{aligned}$$

909. The two brackets multiply to give zero, so at least one of them must be zero. This gives the locus as a pair of intersecting lines, $x = 2$ and $y = 3$:



910. Let a be the length of the diagonal which is a line of symmetry, and b the perpendicular diagonal. The kite consists of two symmetrical triangles with base a and perpendicular height $\frac{1}{2}b$. The area of each is $\frac{1}{2} \cdot a \cdot \frac{1}{2}b$, which is $\frac{1}{4}ab$. Doubling this, the area of the kite is $A = \frac{1}{2}ab$. QED.

911. The roots of the quadratic factor are $x = \pm a$. So, we need only test whether either of these are roots of the cubic factor: $x = a$ is not, but $x = -a$ is. So $x = -a$ is the double root of the equation.

912. The numerator has roots of $x = \pm\sqrt{3}$, from the factor $(x^2 - 3)$. It has no more than this, since $(x^2 + 2)$ is never zero. The solution will therefore be $x = \pm\sqrt{3}$, unless these values are also roots of the denominator. This is the case if $b = -3$, for which the equation has no real roots.

913. The interior angles must sum to 2π, so their mean is $\pi/2$. The smallest and largest angles must be symmetrically either side of this. The lower bound (excluded) is 0; the upper bound (included) is $\pi/2$. So, $\theta \in (0, \pi/2]$.

— Nota Bene —

In the above, by including the bound $\theta = \pi/2$, we are allowing the constant AP case, in which all of the angles are $\pi/2$, i.e. in which the quadrilateral is a square. A constant sequence is an AP with common difference 0. It is also, incidentally, a GP with common ratio 1.

914. (a) Kepler's third law, in the form we need, is that $r = kT^{\frac{2}{3}}$. Substituting $T = 1$ and $r = 152$, we get $k = 152$ (3sf). So, $r = 152 \times T^{\frac{2}{3}}$.

(b) Substituting $T = 2$ gives an orbital radius of 241 million km (3sf).

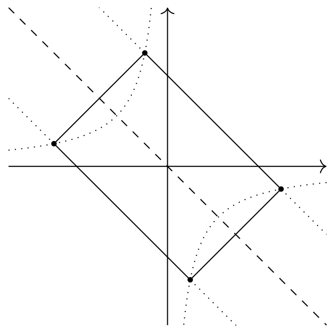
915. The logarithmic equation $\log_9 y = x$, written as an index equation, is $y = 9^x$. Taking the positive square root,

$$\sqrt{y} = \sqrt{9^x} \equiv \left(9^{\frac{1}{2}}\right)^x \equiv 3^x.$$

So, $3^x = \sqrt{y}$.

916. The line $y = -x$ is a line of symmetry of $xy = -5$, since replacing x with $-y$ and replacing y by $-x$ produces $-y \cdot -x = -5$, which is the same graph $xy = -5$.

So, under reflection in $y = -x$, the points on $x + y = 4$ are taken to the points on $x + y = -4$, which guarantees that the quadrilateral has four right-angles, and is therefore a rectangle.

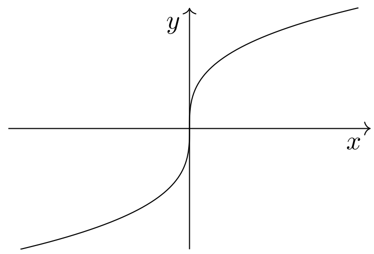


917. (a) The largest possible domain is the set of cubic equations with real coefficients, or equivalently the set of quadruples $(a, b, c, d) \in \mathbb{R}^4$, where $a \neq 0$.

(b) Suitable codomains are $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$, or any set containing $\{1, 2, 3\}$.

(c) A cubic equation must have at least one real root. It can have a maximum of three roots. So, the range is $\{1, 2, 3\}$.

918. Cubing both sides, we have $y^3 = x$, which is the graph $y = x^3$ reflected in the line $y = x$:



919. The possibility space is

	1	2	3	4	5	6
1	✓					
2		✓				
3			✓			
4				✓		

So, the probability is $\frac{4}{24} = \frac{1}{6}$.

— Alternative Method —

Imagine rolling the four-sided die first. This gives a result in $\{1, 2, 3, 4\}$. Each of these outcomes is in the possibility space $\{1, 2, 3, 4, 5, 6\}$ for the six-sided die, so the probability of success is $\frac{1}{6}$.

920. Since the numerator consists of two symmetrical terms, the even-powered terms will cancel. Using the binomial expansion, the numerator is

$$\begin{aligned} &(2^x + 1)^3 + (2^x - 1)^3 \\ \equiv\ & 2 \cdot (2^x)^3 + 6 \cdot 2^x \\ \equiv\ & 2 \cdot 2^{3x} + 6 \cdot 2^x \\ \equiv\ & 2^x(2 \cdot 2^{2x} + 6) \end{aligned}$$

Dividing this by 2^x, the original fraction is

$$2 \cdot 2^{2x} + 6 \equiv 2^{2x+1} + 6.$$

921. The equation for intersections is $\mathrm{f}(x) = \mathrm{g}(x)$. The greatest possible power in this equation is $\max(m, n)$. So, this is a polynomial equation of degree at most $\max(m, n)$. And a polynomial of degree k has at most k roots. Hence, there can be at most $\max(m, n)$ points of intersection. QED.

922. Labelling the bottom-right corner A and calling the side length of the square 1, the equations are

$$\begin{aligned} \updownarrow &: \tfrac{\sqrt{2}}{2}H - 10 = 0 \\ \leftrightarrow &: \tfrac{\sqrt{2}}{2}H + G - F = 0 \\ \overset{\frown}{A} &: G - \tfrac{1}{2} \cdot 10 = 0. \end{aligned}$$

Hence, $G = 5$, $H = 10\sqrt{2}$, and lastly $F = 15$.

— Nota Bene —

You could, in the above, choose to take moments around any point. The set of equations would end up collectively equivalent. However, the bottom-right corner is best, as it is a point of intersection of two of the lines of action. In such statics problems, it's worth taking time to ensure that you've chosen the best point around which to take moments.

923. To find the point at which the gradient is 456, we set the derivative to 456:

$$\begin{aligned} &\tfrac{d}{dx}(x^3 + x^2) = 456 \\ \Longrightarrow\ & 3x^2 + 2x - 456 = 0. \end{aligned}$$

This has roots $x = 12$ and $x = -38/3$. Testing the y values, we see that $(12, 1872)$ is on both the curve and the line, and is therefore a point of tangency.

— Alternative Method —

The equation for intersections is

$$\begin{aligned} & x^3 + x^2 = 456x - 3600 \\ \Longrightarrow\ & x^3 + x^2 - 456x + 3600 = 0. \end{aligned}$$

Using a polynomial solver, we find that there is a root at $x = 12$. Taking out the factor of $(x - 12)$,

$$\begin{aligned} & x^3 + x^2 - 456x + 3600 = 0 \\ \Longrightarrow\ & (x - 12)(x^2 + 13x - 300) = 0 \\ \Longrightarrow\ & (x - 12)^2(x + 25) = 0. \end{aligned}$$

This has a double root at $x = 12$. So, the line is tangent to the curve at $x = 12$.

924. Squaring the equations,

$$\begin{aligned} R^2 \sin^2\theta &= 36, \\ R^2 \cos^2\theta &= 64. \end{aligned}$$

Adding these,

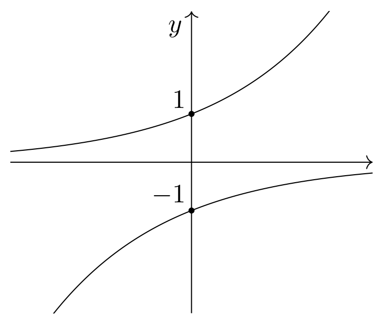
$$\begin{aligned} & R^2(\sin^2\theta + \cos^2\theta) = 100 \\ \Longrightarrow\ & R^2 = 100 \\ \Longrightarrow\ & R = \pm 10. \end{aligned}$$

925. We use the vector **suva**t equation $\mathbf{s} = \mathbf{u}t + \frac{1}{2}\mathbf{a}t^2$. Converting into column vectors for ease,

$$\begin{aligned} \mathbf{s} &= \begin{pmatrix} 5 \\ 3 \end{pmatrix} \times 5 + \tfrac{1}{2}\begin{pmatrix} 2 \\ -4 \end{pmatrix} \times 5^2 \\ &= \begin{pmatrix} 50 \\ -35 \end{pmatrix}. \end{aligned}$$

Dividing by the duration 5, the average velocity is $\bar{\mathbf{v}} = 10\mathbf{i} - 7\mathbf{j}\ \mathrm{ms}^{-1}$.

926. The first is a standard exponential. The second graph has undergone two reflections, in x and y, which amounts to rotation 180° around the origin:



927. Since the square is symmetrical, we can place the 1 without loss of generality. Having placed the 1, there are 6 possible arrangements of the remaining three integers, only one of which yields ascending order when read clockwise. So, $p = \frac{1}{6}$.

928. Completing the square, the circle has equation

$$\left(x+1\right)^2 + \left(y - \tfrac{3}{2}\right)^2 = \tfrac{5}{4}.$$

Solving simultaneously, the intersection of the lines is at $\left(-\frac{6}{5}, \frac{11}{5}\right)$. To test this, we evaluate the LHS of the circle equation at this point, which gives

$$\left(x+1\right)^2 + \left(y - \tfrac{3}{2}\right)^2 \Big|_{\left(-\frac{6}{5}, \frac{11}{5}\right)} = 0.53 < \tfrac{5}{4}.$$

Hence, the lines intersect inside the circle.

929. (a) If f satisfies $\mathrm{f}(-x) \equiv \mathrm{f}(x)$, then the output is the same at x and $-x$. This means that the line $x = 0$ is a line of symmetry. $\mathrm{f}(x) = x^2$. This is known as even symmetry.

(b) If f satisfies $\mathrm{f}(-x) \equiv -\mathrm{f}(x)$, then the output at $-x$ is the negative of the output at x. This means that the curve $y = \mathrm{f}(x)$ has rotational symmetry, order 2, around the origin. This is known as odd symmetry.

— Nota Bene —

Many elementary functions are either even or odd. Most composite functions are neither.

Symmetry	Examples
Even	$x^2, x^6 + 2x^4, \cos x, e^{\lvert x\rvert}$
Odd	$x^3, x^5 - x^3 + 2x, \sin x, \tan x$
Neither	$x^2 + x, e^x, \sin x + 2, \log_2 x$

930. This is a quadratic in a^2b. Completing the square,

$$\begin{aligned} &5a^4b^2 + 100a^2b + 1 \\ \equiv\ &5(a^2b + 10)^2 - 500 + 1 \\ \equiv\ &5(a^2b + 10)^2 - 499. \end{aligned}$$

931. The probabilities are the same. Since the order of king-then-queen is fixed, and since the cards are chosen with replacement, there is no difference mathematically between the two cases: in each, the probability is $\frac{1}{13} \times \frac{1}{13}$.

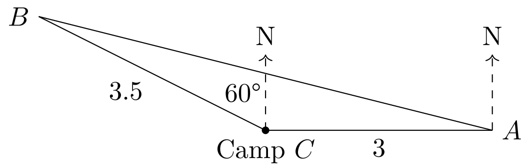
932. In an equilateral triangle of side length l, height is $\frac{\sqrt{3}}{2}l$, so the area is given by

$$A_\triangle = \frac{\sqrt{3}}{4}l^2.$$

The perimeter is $P = 3l$, so $l = \frac{1}{3}P$. We sub this into the formula for the area:

$$\begin{aligned} A_\triangle &= \frac{\sqrt{3}}{4}l^2 \\ &= \frac{\sqrt{3}}{4}\left(\tfrac{1}{3}P\right)^2 \\ &\equiv \frac{\sqrt{3}P^2}{36}, \text{ as required.} \end{aligned}$$

933. After one hour, the situation is as follows:



The distance AB is given by

$$\begin{aligned} &AB^2 = 3^2 + 3.5^2 - 2 \cdot 3 \cdot 3.5 \cos 150 \\ \Longrightarrow\ &AB = 6.27985... \end{aligned}$$

Then the sine rule tells us that

$$\begin{aligned} CAB &= \arcsin\left(\frac{3.5 \sin 150}{6.27985}\right) \\ &= 16.180...^\circ. \end{aligned}$$

So, the bearing of B from A is $270 + 16.2 = 286.2°$ (1dp). This bearing is constant, as varying the time $t > 0$ will scale the entire triangle ABC, and similar triangles contain the same angles.

934. Quadrilateral $BXDY$ is a square. It is congruent to $ABCD$: by symmetry, its sides are equal and its angles are equal.

935. To produce the term in t^2, we need $3(t-1)^2$. This gives a by-product of $-6t$ (and a constant term). So, we need $10(t-1)$ to produce the term in t. The constant term is then $+1$, which yields

$$3t^2 + 4t - 6 \equiv 3(t-1)^2 + 10(t-1) + 1.$$

— Alternative Method —

Let $x = t - 1$, so that $t = x + 1$. Substituting,

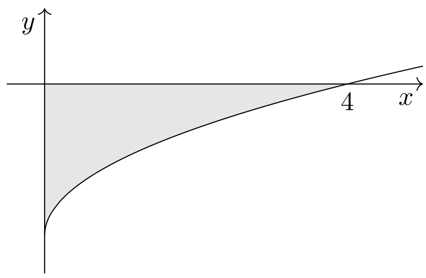
$$\begin{aligned} &3t^2 + 4t - 6 \\ =\ &3(x+1)^2 + 4(x+1) - 6 \\ \equiv\ &3x^2 + 10x + 1 \\ =\ &3(t-1)^2 + 10(t-1) + 1. \end{aligned}$$

936. With the modern usage of the word "reaction", which is not Newton's original usage, this is true by definition.

- Reaction forces are components of contact forces which act perpendicular to the surfaces in contact.

- Frictional forces are components of contact forces which act parallel to the surfaces in contact.

937. Sketching, the scenario is



The signed area is negative, because the curve is below the x axis. So, we include a negative sign to produce a positive area:

$$\begin{aligned} A &= -\int_0^4 \sqrt{x} - 2\,dx \\ &= -\left[\tfrac{2}{3}x^{\frac{3}{2}} - 2x\right]_0^4 \\ &= -\tfrac{2}{3}\cdot 4^{\frac{3}{2}} + 2\cdot 4 \\ &= \tfrac{8}{3}. \end{aligned}$$

938. If $\mathrm{f}'(x)$ is given, then, by integrating, $\mathrm{f}(x)$ is given, up to an additive constant $+c$. In the expression $\mathrm{f}(p) - \mathrm{f}(q)$, the additive constant cancels, but in the expression $\mathrm{f}(p) + \mathrm{f}(q)$ it does not. Hence, the former can be evaluated, but not the latter. $\square$

939. For the shading to form one region, both tiles must have shading on the common edge. For each, the probability of this is $\frac{1}{2}$. So, the probability that both do is $p = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

940. (a) Yes. The sum is equal to the mean multiplied by the number of terms.
   (b) No. The combination of first and last term gives the mean, and this set of information does not give the number of terms.
   (c) Yes. The formula is $S_n = \frac{1}{2}(a + l)n$.

941. At 5:40, the minute hand is $4\pi/3$ radians clockwise from 12. The hour hand is $17\pi/18$ radians clockwise from 12. So, the required angle is $7\pi/18$ radians.

942. Taking out the common factor $(2n - 1)$,

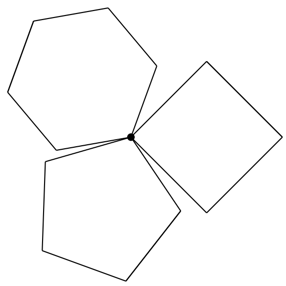
$$\begin{aligned} &2n^2(2n-1) - (2n-1)^2 = 0 \\ \Longrightarrow\ &(2n-1)(2n^2 - (2n-1)) = 0 \\ \Longrightarrow\ &(2n-1)(2n^2 - 2n + 1) = 0. \end{aligned}$$

The quadratic factor has $\Delta = -4 < 0$, so the only root is $n = \frac{1}{2}$.

943. In each of these, we can simply add the indices. f^{-1} means “undo f”, while f^2 means “apply f twice”. In each case, the index means “the number of times f is applied”. This gives
   (a) $\mathrm{f}^{-1}\mathrm{f}\,\mathrm{f}^{-1}(x) \equiv \mathrm{f}^{-1}(x)$,
   (b) $\mathrm{f}^{-1}\mathrm{f}^2(x) \equiv \mathrm{f}(x)$.

944. The first sentence is the problem: it is imprecise to say that friction always acts to oppose motion. It is true that *dynamic* friction always acts to oppose motion, but *static* friction acts to oppose potential motion, i.e. motion that would take place were the friction not present. When a car is parked on a hill, it is static friction on the tyres that stops the car sliding downhill.

945. The minimal case, with the least angle exterior to the three shapes, occurs when the polygons do not overlap.



In this case, the angle exterior to all three shapes is given, in degrees, by

$$360 - \left(180 - \tfrac{360}{4}\right) - \left(180 - \tfrac{360}{5}\right) - \left(180 - \tfrac{360}{6}\right).$$

This is 42°. Hence, if overlaps are considered, at least 42° of angle must be exterior to all three shapes, as required.

946. (a) The integrand is a formula for the velocity at time t. So, the acceleration is $\frac{2}{5}$. The time duration is $\Delta t = 6 - 2 = 4$.
   (b) Evaluating the velocity at $t = 2$ and $t = 6$,

$$\begin{aligned} u &= 1 + \tfrac{2}{5}t\Big|_{t=2} = 1.8, \\ v &= 1 + \tfrac{2}{5}t\Big|_{t=6} = 3.4. \end{aligned}$$

   (c) Carrying out the integration,

$$\begin{aligned} s &= \int_2^6 1 + \tfrac{2}{5}t\,dt \\ &= \left[t + \tfrac{1}{5}t^2\right]_2^6 \\ &= 10.4. \end{aligned}$$

   (d) Using $s = ut + \frac{1}{2}at^2$,

$$s = 1.8\cdot 4 + \tfrac{1}{2}\cdot\tfrac{2}{5}\cdot 4^2 = 10.4.$$

947. When the parameter λ changes by 1, the point (x, y) moves by vector $6\mathbf{i} + 8\mathbf{j}$. By Pythagoras, this has length 10. So, over the interval $[-1, 1]$, the displacement is 20.

—— Alternative Method ——

Substituting $\lambda = -1$ and $\lambda = 1$, the endpoints of the line segment are $(-3, -9)$ and $(9, 7)$. The length of the line segment is the distance between these. By Pythagoras, $s = \sqrt{12^2 + 16^2} = 20$.

948. There are many reasons why this is incorrect. Two reasons are as follows:

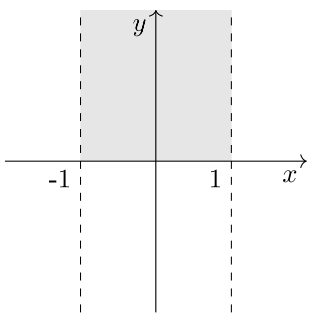
- The experiment does not consist of a set of Y/N trials, which is necessary for the use of a binomial model.
- The scores 0 and 1 are not attainable in the experiment, but have non-zero probability in the proposed model $B(12, 1/6)$.

949. For equilibrium, the resultant force in both x and y directions must be zero. So, we set up and simplify simultaneous equations:

$$x : a + 2b + 1 = 0$$
$$y : 3(a + 2b) + 4 = 0.$$

These have no solutions, as the variable $a+2b$ can not be simultaneously -1 and $-\frac{4}{3}$. Hence, there are no values of a and b for which the object will remain in equilibrium.

950. The boundary equations are $|x| = 2$, which is the pair of lines $x = \pm 2$, and $y = 0$. Using dashed lines for strict inequality, the region is as follows:



951. (a) "Every square is a rectangle." This is true by definition.

(b) "Not all rectangles are parallelograms." This is false. Every rectangle is also a parallelogram.

(c) "Some kites are trapezia." True. For example, a square is both kite and trapezium.

952. Substituting the definition of the function f,

$$\begin{aligned}&\int_0^{\mathrm{f}(a)} \mathrm{f}(x)\,dx\\ =\ &\int_0^{a-1} x - 1\,dx\\ =\ &\left[\tfrac{1}{2}x^2 - x\right]_0^{a-1}\end{aligned}$$

Evaluating this, the equation is quadratic:

$$\tfrac{1}{2}(a-1)^2 - (a-1) = 0.$$

We can take out a factor of $(a-1)$ directly, giving

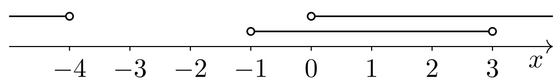
$$\begin{aligned}&(a-1)(a-3) = 0\\ \Longrightarrow\ &a = 1, 3.\end{aligned}$$

953. Differentiating to find the gradient formula,

$$\begin{aligned}&y = x^{\frac{1}{2}} + 4x^{-\frac{1}{2}}\\ \Longrightarrow\ &\frac{dy}{dx} = \tfrac{1}{2}x^{-\frac{1}{2}} - 2x^{-\frac{3}{2}}.\end{aligned}$$

Evaluating at $x = 1$ gives a gradient of $-3/2$. So, the tangent has equation $3x+2y = k$. Substituting $(1, 5)$ gives $k = 13$, which is the required result.

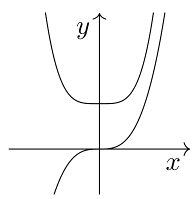
954. The latter set is all x values which differ from -2 by more than 2. This is $(-\infty, -4) \cup (0, \infty)$. The intersection of this set with $(-1, 3)$ is the interval $(0, 3)$.



955. (a) Yes, the curves do intersect. The equation for intersections is cubic, $x^3 = x^2 + 1$, and a cubic equation always has a real root.

(b) No, the curves do not intersect. The curves are vertical translations of one another.

(c) No, the curves do not intersect. For $x < 1$, there can be no intersection because $x^3 < 1$ and $x^4 + 1 > 1$. And for $x \geq 1$, there can't be intersection because $x^4 \geq x^3$, so $x^4 + 1 > x^3$.

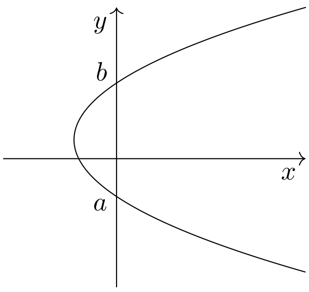


956. The largest angle is opposite the longest side. So, using the cosine rule,

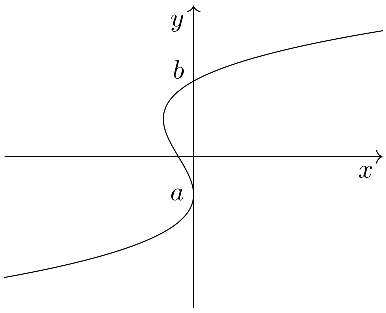
$$\begin{aligned}\theta &= \arccos\left(\frac{10^2 + 12^2 - 15^2}{2 \cdot 10 \cdot 12}\right)\\ &= 1.49154...\\ &\approx 1.5 \text{ rad, as required.}\end{aligned}$$

957. The possibility space is originally $\{\text{HH}, \text{HT}, \text{TH}, \text{TT}\}$. The condition that at least one head is observed restricts this, ruling out TT. The possibility space is now $\{\text{HH}, \text{HT}, \text{TH}\}$. Hence, the probability is $1/3$.

958. (a) $x = (y - a)(y - b)$ is a positive parabola with y intercepts at $y = a$ and $y = b$:



(b) $x = (y - a)^2(y - b)$ is a positive cubic with a double root (just touches) at $y = a$ and a single root (crosses) at $y = b$:



959. We want $16\mathbf{i} + 25\mathbf{j} = p\mathbf{a} + q\mathbf{b}$, for some $p, q \in \mathbb{R}$. Expanding this and equating coefficients,

$$\begin{aligned} \mathbf{i} &: 16 = 4p + 2q \\ \mathbf{j} &: 25 = 3p - 5q. \end{aligned}$$

Solving simultaneously gives $p = 5, q = -2$, so $16\mathbf{i} + 25\mathbf{j} = 5\mathbf{a} - 2\mathbf{b}$.

960. The curve is a circle with centre $(1, 0)$ and radius 5. The normal is perpendicular to the tangent, so it must be a radius. Hence, the normal at $(4, 4)$ passes through $(1, 0)$.

The gradient is $4/3$. The equation of the normal is therefore $y = \frac{4}{3}(x - 1)$. In the required form, with all coefficients integers, this is $4x - 3y - 4 = 0$.

961. The first difference is given by

$$\begin{aligned} d_n &= u_{n+1} - u_n \\ &= \big(a(n+1)^2 + b(n+1) + c\big) - \big(an^2 + bn + c\big) \\ &= 2an + a + b. \end{aligned}$$

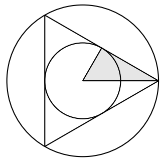
So, d_n is an AP. Increasing n by one increases the value of d_n by $2a$, so the second difference is a constant $2a$. QED.

962. The fixed points of a function g are unaffected by the application of the function, i.e. they satisfy $g(x) = x$. Setting up the relevant equation,

$$\begin{aligned} & x^2 + 4x + 2 = x \\ \Longrightarrow\ & x^2 + 3x + 2 = 0 \\ \Longrightarrow\ & x = -1, -2. \end{aligned}$$

So, g has two fixed points.

963. (a) The diagram is

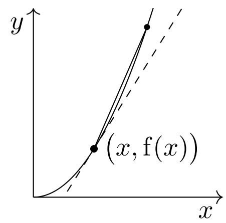


The ratio $a : R$ is opposite to hypotenuse in the right-angled triangle shown. So, $a : R$ is $\sin 30° : 1$, which is $1 : 2$.

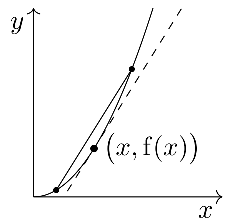
(b) The same argument holds, this time bisecting the right angle of the square to give $\sin 45° : 1$, which is $1 : \sqrt{2}$.

(c) Again, $a : R$ is $\sin 60° : 1$, which is $1 : \frac{\sqrt{3}}{2}$.

964. (a) The numerator and denominator are δy and δx, small but finite lengths in y and x. They are the sides of the gradient triangle of a short chord. That chord is constructed from x to $x + h$, to the *right* of the x value at which the tangent gradient is sought.



(b) The numerator and denominator are still δy and δx, sides of a gradient triangle. In this version, however, the chord is set up between points *either side* of the x value at which the tangent gradient is sought. These points are $\big(x - h, f(x - h)\big)$, to the left of x, and $\big(x + h, f(x + h)\big)$ to the right of x.



Both points then tend to x as h tends to zero. The infinitesimal limit is the same, as can be verified by simplifying the algebra of the limit.

965. The boundary values are $x = 4$ and $x = 5$, so these must be the roots of the boundary equation $x^2 + px + q = 0$. The inequality must therefore be $(x-4)(x-5) > 0$. Multiplying this out, $p = -9$ and $q = 20$.

966. Using the factorial definition of ${}^n\mathrm{C}_r$,

$$\begin{aligned} {}^n\mathrm{C}_1 + {}^n\mathrm{C}_2 &\equiv \frac{n!}{1!(n-1)!} + \frac{n!}{2!(n-2)!} \\ &\equiv n + \frac{n(n+1)}{2} \\ &\equiv \frac{n^2+3n}{2}, \text{ as required.} \end{aligned}$$

967. The RHS has a double factor of $(x+b)$ and a triple factor of $(x+c)$, so it has a double root at $x = -b$ and a triple root at $x = -c$. The curve crosses the x axis at $x = -4$, so this must be the triple root. This gives $c = 4$ and $b = -1$. Then, substituting the value $x = 0$ yields $64a = 16$, so $a = 4$.

— Nota Bene —

At a root of odd multiplicity, there is a sign change. At a root of even multiplicity, however, there is no sign change.

968. (a) $\mathbb{P}(A \mid A) = 1$. If A is given, then occurrence of A is certain.
(b) $\mathbb{P}(A \cap B' \mid B) = 0$. If B is given, then B' cannot have occurred.
(c) $\mathbb{P}(A \cap B \mid A' \cup B') = 0$. Events $A \cap B$ and $A' \cup B'$ are complementary and thus mutually exclusive. If the latter has occurred, then the former cannot have done.

969. Setting $x^6 + x^3 + 1 = 0$, we have a quadratic in x^3. Its discriminant is $\Delta = 1 - 4 = -3 < 0$, so no values of x^3 satisfy this equation. Hence, no values of x do, and the curve does not cross the x axis.

970. (a) The range is translated by a, giving $[0, 2a]$.
(b) The range is translated by $-a$, giving $[-2a, 0]$.
(c) This is the negative of the function in (b), so the range is $[0, 2a]$.

971. Expanding the numerator gives $24x^3 + 36x^2 + 18x$. We can then cancel a factor of x. This gives

$$\begin{aligned} &\int_2^4 24x^2 + 36x + 18\,dx \\ &= \Big[8x^3 + 18x^2 + 18x\Big]_2^4 \\ &= (8\cdot 4^3 + 18\cdot 4^2 + 18\cdot 4) \\ &\qquad - (8\cdot 2^3 + 18\cdot 2^2 + 18\cdot 2) \\ &= 700, \text{ as required.} \end{aligned}$$

972. (a) Differentiating,

$$y = \frac{1}{x} \implies \frac{dy}{dx} = -\frac{1}{x^2}.$$

Evaluating at $x = a$ gives $-1/a^2$, which is why this appears as the gradient of the tangent line.

(b) Substituting the point $(a, 1/a)$ gives

$$\begin{aligned} \frac{1}{a} &= -\frac{1}{a^2}a + c \\ \implies c &= \frac{2}{a}. \end{aligned}$$

This is the required result.

973. The terms of the sum are

i	$(1+4i-3i^2)x^i$
0	1
1	$2x$
2	$-3x^2$

So, writing the sum out longhand,

$$\begin{aligned} &1 + 2x - 3x^2 = 0 \\ \implies &(1-x)(1-3x) = 0 \\ \implies &x = \tfrac{1}{3}, 1. \end{aligned}$$

974. Multiplying the first by $\sqrt{3}$, the equations are

$$\begin{aligned} \sqrt{3}x &= 3a - \sqrt{3}b, \\ y &= a + \sqrt{3}b. \end{aligned}$$

Adding these eliminates b:

$$\begin{aligned} &\sqrt{3}x + y = 4a \\ \implies &a = \tfrac{1}{4}(\sqrt{3}x + y). \end{aligned}$$

Substituting in, $b = \frac{1}{4}(\sqrt{3}y - x)$.

975. To end in zero, a number requires a factor of 10, so factors of 2 and 5. A product of five consecutive integers necessarily has both. A product of four does not, however: $4! = 24$ is a counterexample. Hence, $n_{\min} = 5$.

976. By Pythagoras, $\triangle ABC$ has lengths $\big(1, \sqrt{2}, \sqrt{3}\big)$. Furthermore, it is a right-angled triangle. Basic trig gives $\angle ACB = \arcsin 1/\sqrt{3}$.

977. The numerator is a quadratic. So, for the fraction to be expressible as a linear function of x, $x - 6$ must be a factor of the numerator. So, $x = 6$ must be a root, giving $36 + 6p + 2 = 0$. Hence, $p = -\frac{38}{6} = -\frac{19}{3}$.

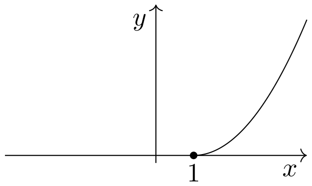
978. The boat will go backwards.

While the blown air will exert a force forwards on the sail, this cannot exceed the backwards force exerted on the fan by the air. NIII guarantees this. If the sail catches the air perfectly, the two effects will cancel out, leaving the boat in equilibrium. In practice, this won't be the case, and the forwards force on the sail will be less than the backwards force on the fan.

— Nota Bene —

It is possible to mimic a "perfect sail" by placing the fan inside the cabin. If sealed inside, the fan will not push the boat forwards or backwards.

979. We can square both sides to give $y = (x-1)^2$, but this will only be valid for $x - 1 \geq 0$. So, the graph is half of a parabola:



980. The possibility space is

	1	2	3	4	5	6
1	✓	✓				
2	✓	✓	✓			
3		✓	✓	✓		
4			✓	✓	✓	
5				✓	✓	✓
6					✓	✓

So, $p = \frac{16}{36} = \frac{4}{9}$.

981. We are told that, for $a \neq 0$,

$$\int \mathrm{f}(x) + \mathrm{g}(x)\, dx = ax^2 + bx + c.$$

Differentiating both sides with respect to x gives

$$\mathrm{f}(x) + \mathrm{g}(x) = 2ax + b.$$

So, the equation $\mathrm{f}(x) + \mathrm{g}(x) = 0$ is $2ax + b = 0$. Since $a \neq 0$, this will always have exactly one root $x = -{}^{b}\!/_{2a}$.

982. The implication holds: each side is an equivalent definition of independence. To prove it, we begin with the right-hand equation, using the definition $\mathbb{P}(A \cap B) = \mathbb{P}(A \mid B)\,\mathbb{P}(B)$:

$$\begin{aligned} & \mathbb{P}(A)\,\mathbb{P}(B) = \mathbb{P}(A \cap B) \\ \iff & \mathbb{P}(A)\,\mathbb{P}(B) = \mathbb{P}(A \mid B)\,\mathbb{P}(B) \\ \iff & \mathbb{P}(A) = \mathbb{P}(A \mid B). \end{aligned}$$

This is the required result.

— Nota Bene —

There is a division by $\mathbb{P}(B)$ in the above, from the second to the third line. However, $\mathbb{P}(B) = 0$ is not a relevant concern, as conditional probability based on B is not defined when $\mathbb{P}(B) = 0$.

983. The instruction is not defined when $x \in \{2, 3\}$. Since neither of those values is in the domain $[0, 1]$, the function h is well defined as it is.

984. Expressing this algebraically,

$$\begin{aligned} & a(a+8) = -7 \\ \implies & a^2 + 8a + 7 = 0 \\ \implies & a = -7, -1. \end{aligned}$$

The second term is $a + 4$, which has value ± 3.

985. The sum of two consecutive odd squares may be expressed algebraically as

$$\begin{aligned} & (2n+1)^2 - (2n-1)^2 \\ \equiv\ & 4n^2 + 4n + 1 - (4n^2 - 4n + 1) \\ \equiv\ & 8n. \end{aligned}$$

Since $n \in \mathbb{Z}$, this is divisible by 8. QED.

986. (a) Region R contains all (x, y) points satisfying both $4x^2 - 2x \leq y$ and $y \leq x^2$. The boundary equations are $y = 4x^2 - 2x$ and $y = x^2$. The equation for intersections is $x^2 = 4x^2 - 2x$, which gives $x = 0, \frac{2}{3}$. So, $k = \frac{2}{3}$.

(b) The integrand is the y distance between the parabolae: $x^2 - (4x^2 - 2x) \equiv 2x - 3x^2$.

(c) Carrying out the integral,

$$\begin{aligned} A_R &= \int_0^{\frac{2}{3}} 2x - 3x^2\, dx \\ &= \Big[x^2 - x^3\Big]_0^{\frac{2}{3}} \\ &= \left(\tfrac{2}{3}\right)^2 - \left(\tfrac{2}{3}\right)^3 \\ &= \tfrac{4}{27}. \end{aligned}$$

987. (a) There are 64 squares on which to place a set of 8 identical pawns. So, there are ${}^{64}\mathrm{C}_8$ ways.

(b) As in (a), there are ${}^{64}\mathrm{C}_8$ ways of placing the white pawns, and then ${}^{56}\mathrm{C}_8$ ways of placing the black. This gives ${}^{64}\mathrm{C}_8 \times {}^{56}\mathrm{C}_8$ ways in total.

— Alternative Method —

First choosing 16 squares on which to place the 16 pawns, and then choosing 8 for the white pawns, the result above may also be expressed as ${}^{64}\mathrm{C}_{16} \times {}^{16}\mathrm{C}_8$.

988. The gradients of the vectors, which are a_2/a_1 and b_2/b_1, must multiply to give -1:

$$\begin{aligned} &\frac{a_2}{a_1}\cdot\frac{b_2}{b_1} = -1 \\ \Longrightarrow\ & a_2b_2 = -a_1b_1 \\ \Longrightarrow\ & a_1b_1 + a_2b_2 = 0, \text{ as required.} \end{aligned}$$

989. There are two transformations here.

- *Inputs*: x has been replaced by $2x$. This is a stretch by scale factor $\frac{1}{2}$ in the x direction.
- *Outputs*: the outputs have been multiplied by 2. This is a stretch by scale factor 2 in the y direction.

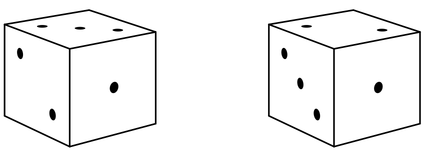
990. Scaling and rearranging, the equations are

$$\begin{aligned} 4x + 6y &= 14 \\ 4x + 6y &= 1. \end{aligned}$$

These are a pair of distinct parallel lines, and do not intersect. Hence, S is the empty set, which means that $\text{n}(S) = 0$.

991. The lower bounds satisfy $k-4 < k-1$. The upper bounds satisfy $k+3 < k+6$. And the intervals always overlap, since $k-1 < k+3$. Hence, the interval simplifies to $(k-1, k+3]$.

992. Numbers $1, 2, 5, 6$ may be placed without loss of generality, as any placement may be rotated into any other. This leaves two opposite space faces, for the 3 and the 4. There are two choices, which are not equivalent. Numbers $1, 2, 3$ may be arranged in one of the following two configurations.



These cannot be rotated one onto the other.

993. Using the cosine rule, the angle between the sides of length 4 and 5 is given by

$$\cos\theta = \frac{6^2 - 4^2 - 5^2}{2\cdot 4\cdot 5} = -\tfrac{1}{8}.$$

The first Pythagorean trig identity gives

$$\sin\theta = \sqrt{1 - \tfrac{1}{64}} = \tfrac{3\sqrt{7}}{8}.$$

Substituting into $A_\triangle = \frac{1}{2}ab\sin C$,

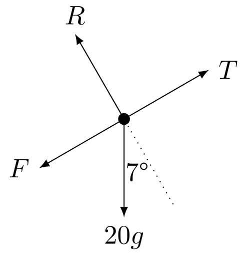
$$\begin{aligned} A_\triangle &= \tfrac{1}{2}\cdot 4\cdot 5\cdot \tfrac{3\sqrt{7}}{8} \\ &= \tfrac{15\sqrt{7}}{4}. \end{aligned}$$

— Alternative Method —

The semiperimeter is $s = \frac{1}{2}(4+5+6)$, which is $s = \frac{15}{2}$. Heron's formula gives

$$\begin{aligned} A_\triangle &= \sqrt{s(s-a)(s-b)(s-c)} \\ &= \sqrt{\tfrac{15}{2}\left(\tfrac{15}{2}-4\right)\left(\tfrac{15}{2}-5\right)\left(\tfrac{15}{2}-6\right)} \\ &= \tfrac{15\sqrt{7}}{4}. \end{aligned}$$

994. Acceleration is zero, and the forces are as follows (the speed is not relevant):



Resolving normal to the slope, $R = 20g\cos 7°$. And resolving parallel to the slope, using $F_{\text{max}} = \mu R$ since the case is in motion,

$$\begin{aligned} &T - \mu R - 20g\sin 7° = 0 \\ \Longrightarrow\ & T - 0.215\cdot 20g\cos 7° - 20g\sin 7° = 0 \\ \Longrightarrow\ & T = 65.7 \text{ N (3sf)}. \end{aligned}$$

995. Completing the square gives $y = (x+1)^2 + 2$, so (p, q) is $(-1, 2)$. The required monic parabola has a vertex at $(-1, -2)$, so its equation, in completed squared form, is $y = (x+1)^2 - 2$. In polynomial form, this is $y = x^2 - 2x + 3$.

— Alternative Method —

To transform the point (p, q) to $(p, -q)$, we reflect in the y axis, which is a replacement of x by $-x$. Applying this as an input transformation, the new equation is

$$\begin{aligned} y &= (-x)^2 + 2(-x) + 3 \\ &= x^2 - 2x + 3. \end{aligned}$$

996. We can treat the y values as a sequence, since x increases linearly. A quadratic sequence must have a constant second difference. The first differences are $2, 4, 4$, which gives second differences of 2 and 0. Since $2 \neq 0$, the relationship cannot be quadratic.

— Alternative Method —

Using $(0, 0)$, any quadratic relationship must be of the form $y = ax^2 + bx$. Substituting $x = 1$ and $x = 2$,

$$\begin{aligned} 2 &= a + b, \\ 6 &= 4a + 2b. \end{aligned}$$

Solving these, $a = b = 1$. So, the relationship would have to be $y = x^2 + x$. Substituting $x = 3$ gives $y = 12 \neq 10$. So, the relationship cannot be quadratic.

997. Of n different objects, there are $n!$ orders: we need to solve $n! = 40320$. There is no inverse factorial function on a calculator, so we test values. This gives $8! = 40320$, so $n = 8$.

998. (a) Multiplying by $x^3 - x^2 = x^2(x-1)$ gives

$$\begin{aligned} & 1 \equiv A(x-1) + Bx^2 \\ \Longrightarrow\ & 1 \equiv Bx^2 + Ax - A. \end{aligned}$$

(b) The coefficients of x^2 and x require $A, B = 0$. This renders the entire RHS zero.

(c) Because x^2 is a repeated factor, the correct form for these partial fractions is

$$\frac{1}{x^3 - x^2} \equiv \frac{A}{x^2} + \frac{B}{x} + \frac{C}{x-1}.$$

999. Using the general formulae for GPs and APs, the nth terms are

$$\begin{aligned} a_n &= 1.01^{n-1} \\ b_n &= n. \end{aligned}$$

Sequence a_n will reach 1000 when

$$\begin{aligned} & 1.01^{n-1} = 1000 \\ \Longrightarrow\ & n - 1 = \log_{1.01} 1000 \\ \Longrightarrow\ & n = 695.22... \end{aligned}$$

So, sequence a_n will attain 1000 first, at $n = 696$ as opposed to $n = 1000$.

1000. This is a quadratic in $x^{0.2}$:

$$\begin{aligned} & 2x^{0.4} + x^{0.2} - 1 = 0 \\ \Longrightarrow\ & (2x^{0.2} - 1)(x^{0.2} + 1) = 0 \\ \Longrightarrow\ & x^{0.2} = \tfrac{1}{2}, -1. \end{aligned}$$

Raising both sides to the power 5 gives $x = \frac{1}{32}, -1$.

——— END OF VOLUME I ———